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A new approach to the transient conduction in a 2-D rectangular fin

YI-HSU JU,† YI-SHYONG CHOU and CHANG-CHUN HSIAO‡
Department of Chemical Engineering, National Taiwan Institute of Technology, Taipei, Taiwan 10772, R.O.C.

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Abstract—The purpose of this work is to develop a method that enables one to reduce the problem of transient conduction in a 2-D rectangular fin to that of a 1-D problem. After applying the perturbation technique and an averaging method, the problem of transient conduction in a 2-D rectangular fin is reduced to that of a 1-D problem. A linear operator method can then be applied to solve the resulting partial differential equation if the associated Sturm–Liouville problem is manageable. Actual calculation of transient temperature profiles in a 2-D rectangular fin confirms the validity of the method developed in this work.

INTRODUCTION

THE RATE of heat transfer between an object and its surrounding fluid depends on several factors including the surface area of the object. The application of an extended surface is a common practice in engineering to increase the rate of heat transfer between an object and its surrounding fluid. One example is the use of a fin as a device to increase the rate of removal of heat generated by the large amount of power consumed by a large computer.

The steady-state heat conduction has been studied by many investigators. Harper and Brown [1] obtained temperature profiles and efficiencies of straight, wedge and annular fins. Gardner [2] extended previous studies to fins of various geometries and obtained efficiency curves which are still in use in engineering design today.

Chapman [3] studied transient heat conduction in an annular fin by using the separation of variables method. Yang [4] obtained temperature distributions at large time for a rectangular fin subject to periodic base temperature change. Suryanarayana [5] investigated transient conduction in a 1-D rectangular fin by using the Laplace transformation method. He also obtained an approximate solution that is valid at small time.

Most research on transient conduction in a fin is based on the assumption that conduction in the transverse direction is much smaller than that in the longitudinal direction thus reducing the problem to that of a 1-D problem. This is true only when the Biot number is small [6, 7]. There has been relatively little research on transient conduction in a 2-D fin. Chu [8] studied the transient conduction in a rectangular fin by using the Laplace transformation method and the separation of variables method. His solution procedure is very tedious and the solution converges very slowly at small time. Chu *et al.* [9] studied the same problem by using the Laplace transformation method and then obtained inverse Laplace transformation using the Fourier series method. There are drawbacks in their solution. Since the series converges very slowly in order to obtain numerical results accurate to three significant figures, hundreds of terms are required [10]. In addition, there is no sound theoretical base to justify that only the real part need be retained in the inverse transformation [11].

In this study, a regular perturbation technique and an averaging method are applied to reduce the problem of transient conduction in a 2-D rectangular fin to that of a 1-D problem. The linear operator method is then used and the solution of the problem becomes very straightforward and simple.

PROBLEM FORMULATION AND SOLUTION

Consider a 2-D rectangular fin as shown schematically in Fig. 1. At $t \leqslant 0$, the fin is in thermal equilibrium with its surrounding fluid. At $t > 0$, it is subject to a step change in base temperature. Under the following assumptions:

(1) all physical properties are constant;
(2) the convective heat transfer coefficient at each side of the fin is constant;
(3) the top surface of the fin is adiabatic;
(4) the thickness of the fin is much smaller than its length;
(5) there is no heat source in the fin;

the energy equation for the fin can be expressed as

† Author to whom correspondence should be addressed.
‡ Present address: Chemical Engineering Department, Mingchi Institute of Technology, Taishan, Taipei Hsien, Taiwan, R.O.C.

NOMENCLATURE

b	thickness of fin	T^*_∞	ambient temperature
C^*_p	heat capacity of fin	T^*_0	temperature at base of fin
D	parameter defined in equation (24)	x^*	longitudinal coordinate
h_0, h_1	convective heat transfer coefficient	x	dimensionless x^*
H_0	$h_0 b/k^*$	y^*	transverse coordinate
H_1	$h_1 b/k^*$	y	dimensionless y^*.
k^*	thermal conductivity of fin		
l	length of fin		
L	differential operator defined in equation (26)		
q^*_x	heat flux in the longitudinal direction		
q_x	dimensionless heat flux in the longitudinal direction		
$\bar{q}_x$	average of q_x, defined in equations (18) and (19)		
q^*_y	heat flux in the transverse direction		
q_y	dimensionless heat flux in the transverse direction		
t^*	time		
t	dimensionless time		
T^*	temperature of fin		
T	dimensionless temperature of fin		
$\bar{T}$	average T, defined in equation (18)		

Greek symbols

α	constant defined in equation (12)
β	integration constant defined in equation (13)
γ	integration constant defined in equation (15)
ε	perturbation parameter, b/l
λ	eigenvalue
ρ^*	density of fin
ϕ	eigenfunction.

Superscripts

$*$	dimensional quantity
$-$	average value
$'$	d/dx.

$$\rho^* C^*_p \frac{\partial T^*}{\partial t^*} + \frac{\partial q^*_x}{\partial x^*} + \frac{\partial q^*_y}{\partial y^*} = 0 \quad (1)$$

$$q^*_x = -k^* \frac{\partial T^*}{\partial x^*} \quad (2)$$

$$q^*_y = -k^* \frac{\partial T^*}{\partial y^*} \quad (3)$$

with boundary and initial conditions

$$q^*_y|_{y^*=0} = h_0(T^*_\infty - T^*|_{y^*=0})$$

$$q^*_y|_{y^*=b} = h_1(T^*|_{y^*=b} - T^*_\infty)$$

$$T^*|_{x^*=0} = T^*$$

$$\left.\frac{\partial T^*}{\partial x^*}\right|_{x^*=l} = 0$$

$$T^*|_{t^*=0} = T^*_\infty.$$

By defining the following dimensionless quantities:

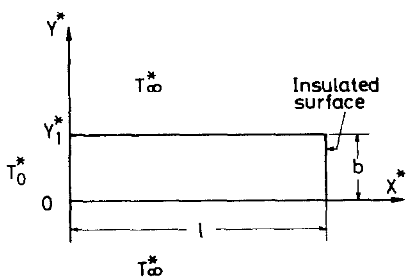

FIG. 1. Schematic diagram of 2-D rectangular fin.

$$\varepsilon = b/l, \quad x = x^*/l, \quad y = y^*/b$$

$$T = \frac{T^* - T^*_0}{T^*_\infty - T^*_0}, \quad t = \frac{k^* t^*}{\rho^* C^*_p l^2}$$

$$q_x = \frac{q^*_x l}{k^*(T^*_\infty - T^*_0)}, \quad q_y = \frac{q^*_y l}{k^*(T^*_\infty - T^*_0)\varepsilon}$$

equations (1)–(3) with the corresponding boundary and initial conditions can be transformed into the following dimensionless form:

$$\frac{\partial T}{\partial t} + \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = 0 \quad (4)$$

$$q_x = -\frac{\partial T}{\partial x} \quad (5)$$

$$\varepsilon^2 q_y = -\frac{\partial T}{\partial y} \quad (6)$$

$$q_y|_{y=0} = \frac{H_0}{\varepsilon^2}(1 - T|_{y=0}) \quad (7)$$

$$q_y|_{y=1} = \frac{H_1}{\varepsilon^2}(T|_{y=1} - 1) \quad (8)$$

$$T|_{x=0} = 0$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=1} = 0$$

$$T|_{t=0} = 1.$$

Since $\varepsilon \ll 1$, it is chosen as the perturbation parameter and each dependent variable is expressed as a power series in ε^2 and substituted in equations (4)–(6). After some algebraic manipulations, we have the zeroth-order equations

$$\frac{\partial T^{\{0\}}}{\partial t}+\frac{\partial q_x^{\{0\}}}{\partial x}+\frac{\partial q_y^{\{0\}}}{\partial y}=0 \tag{9}$$

$$\frac{\partial T^{\{0\}}}{\partial y}=0 \tag{10}$$

$$q_x^{\{0\}}=-\frac{\partial T^{\{0\}}}{\partial x}. \tag{11}$$

From equation (10), we have $T^{\{0\}}=T^{\{0\}}(x,t)$. Similarly, from equation (11), we have $q_x^{\{0\}}=q_x^{\{0\}}(x,t)$. Then

$$\frac{\partial q_y^{\{0\}}}{\partial y}=\alpha \tag{12}$$

as a result of equation (9). Integration of equation (12) yields

$$q_y^{\{0\}}=\alpha y+\beta \tag{13}$$

where $\alpha=\alpha(x,t)$, $\beta=\beta(x,t)$. From the second-order (ε^2) equations, we have

$$q_y^{\{0\}}=-\frac{\partial T^{\{1\}}}{\partial y}. \tag{14}$$

From equation (14), we have

$$T^{\{1\}}=-\left(\frac{\alpha}{2}y^2+\beta y+\gamma\right) \tag{15}$$

where

$$\gamma=\gamma(x,t).$$

Expand $T(x,y,t)$ and $q_y(x,y,t)$ in power series of ε^2 as

$$T(x,y,t)=T^{\{0\}}-\varepsilon^2\left(\frac{\alpha}{2}y^2+\beta y+\gamma\right)+O(\varepsilon^4) \tag{16}$$

$$q_y(x,y,t)=\alpha y+\beta+O(\varepsilon^2). \tag{17}$$

The averages of T and q_x are defined as

$$\bar{T}(x,t)=\int_0^1 T(x,y,t)\,\mathrm{d}y \tag{18}$$

$$\bar{q}_x=-\frac{\partial \bar{T}}{\partial x}. \tag{19}$$

Since the thickness of the fin is usually much smaller than its length, equation (16) or (18) should be a good approximation to the real fin temperature.

After substituting equations (16) and (17) into equation (4) and applying the averaging method as defined in equations (18) and (19), we have

$$\frac{\partial \bar{T}}{\partial t}+\frac{\partial \bar{q}_x}{\partial x}=-\alpha$$

or

$$\frac{\partial \bar{T}}{\partial t}-\frac{\partial^2 \bar{T}}{\partial x^2}=-\alpha \tag{20}$$

where α can be expressed in terms of $\bar{T}$. From equations (16) and (18), we have

$$\bar{T}=T^{\{0\}}-\left(\frac{\alpha}{6}+\frac{\beta}{2}+\gamma\right). \tag{21}$$

Substituting equation (17) into equations (7) and (8) yields

$$\beta=\frac{H_0}{\varepsilon^2}(1-T^{\{0\}}+\varepsilon^2\gamma) \tag{22}$$

$$\alpha+\beta=\frac{H_1}{\varepsilon^2}\left[T^{\{0\}}-\varepsilon^2\left(\frac{\alpha}{2}+\beta+\gamma\right)-1\right]. \tag{23}$$

Equations (21)–(23) can be viewed as three algebraic equations with unknowns α, β and γ. The solution for α is $\alpha=D(\bar{T}-1)$ where

$$D=\frac{12(H_0H_1+H_0+H_1)}{(12+4H_0+4H_1+H_0H_1)}.$$

Equation (20) then becomes

$$\frac{\partial \bar{T}}{\partial t}-\frac{\partial^2 \bar{T}}{\partial x^2}=D\bar{T}-D. \tag{24}$$

Initial and boundary conditions for equation (24) are

$$\bar{T}(x,0)=1$$
$$\bar{T}(0,t)=0$$
$$\left.\frac{\partial \bar{T}}{\partial x}\right|_{x=1}=0. \tag{25}$$

Equation (24) can be rewritten as

$$\frac{\partial \bar{T}}{\partial t}=-L\bar{T}-D \tag{26}$$

where

$$L=-\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}+D\right).$$

The associated Sturm–Liouville problem for equations (25) and (26) is

$$L\phi=\lambda\phi$$
$$\phi(0)=\phi'(1)=0. \tag{27}$$

According to functional analysis theory [12], the associated Sturm–Liouville problem possesses an infinite number of discrete eigenvalues λ_1, λ_2, $\lambda_3,\ldots$ These eigenvalues are all real and positive. There exists an eigenfunction ϕ_j corresponding to each eigenvalue λ_j which is a solution to equation (27). $\bar{T}$ can be expressed as

$$\bar{T}=\sum_{j=1}^{\infty}\frac{\langle\bar{T},\phi_j\rangle}{\langle\phi_j,\phi_j\rangle}\phi_j \tag{28}$$

where the inner product $\langle \cdot, \cdot \rangle$ is defined as

$$\langle \phi, \psi \rangle = \int_0^1 \phi(x)\psi(x)\,dx.$$

Taking the inner product of equation (26) with ϕ_j yields

$$\frac{d\langle \bar{T}, \phi_j \rangle}{dt} = -\lambda_j \langle \bar{T}, \phi_j \rangle + \langle -D, \phi_j \rangle. \quad (29)$$

The initial condition for equation (29) is

$$\langle \bar{T}, \phi_j \rangle = \langle 1, \phi_j \rangle \quad \text{at} \quad t = 0.$$

The solution of equation (29) is

$$\langle \bar{T}, \phi_j \rangle = \langle 1, \phi_j \rangle \exp(-\lambda_j t) + \exp(-\lambda_j t) \int_0^t \langle -D, \phi_j \rangle \exp(\lambda_j z)\,dz. \quad (30)$$

The temperature profile $\bar{T}(x, t)$ can be obtained by substituting equation (30) into equation (28). It remains to determine the eigenvalues and eigenfunction of equation (27). The analytical solution of equation (27) is straightforward and the results are

$$\left.\begin{aligned} \lambda_j &= \left[\frac{(2j-1)\pi}{2}\right]^2 - D \\ \phi_j &= \sin\frac{(2j-1)\pi x}{2} \end{aligned}\right\} \quad j = 1, 2, 3, \ldots$$

After substituting λ_j and ϕ_j into equation (28), $\bar{T}$ can be expressed as

$$\bar{T} = \sum_{j=1}^{\infty} \frac{8}{(4j-1)\pi} \left\{ \exp(-\lambda_{2j} t) + \frac{D}{\lambda_{2j}} [1 - \exp(-\lambda_{2j} t)] \right\} \sin \lambda_{2j} x. \quad (31)$$

RESULTS AND DISCUSSION

From equation (31) it can be seen that the solution converges very fast even for small t. Actual numerical calculations show that in order to obtain results accurate to three significant figures, less than ten terms are needed for $t = 0.01$. Thus the solution obtained in this work converges very fast as compared to previous studies which require hundreds of terms to obtain the same degree of accuracy. Figures 2 and 3 show temperature distributions for Biot numbers of 0.1 and 2.0, respectively, when the fin is subject to a step change in base temperature. These results agree closely with those obtained by Chu *et al.* [9]. When the Biot number is small ($Bi < 1.0$), Chu *et al.*'s results show that there is very little temperature variation in the transverse direction. However, when the Biot number is large, there is some difference between the centre temperature and the surface temperature as can be seen from Fig. 4. The dotted line in Fig. 4 represents results obtained in this study. Figures 5 and 6 show

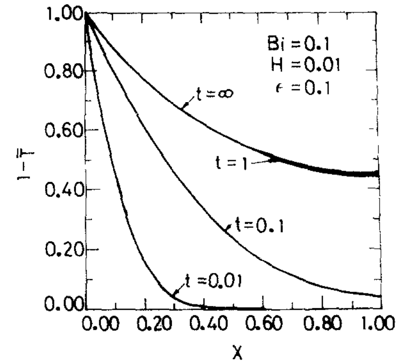

FIG. 2. Temperature profiles of fin for $Bi = 0.1$.

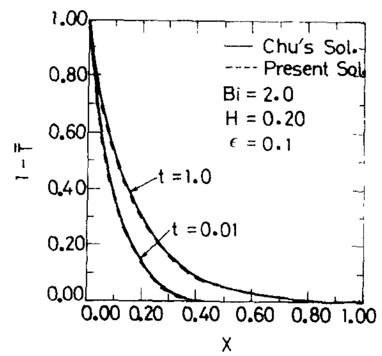

FIG. 3. Temperature profiles of fin for $Bi = 2.0$.

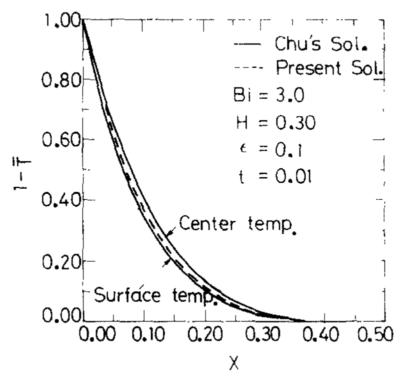

FIG. 4. Comparison of surface, centre and average temperature profiles for $Bi = 3.0$.

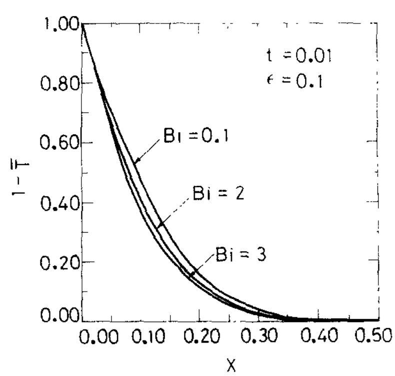

FIG. 5. Effects of Bi on the temperature profiles at $t = 0.01$.

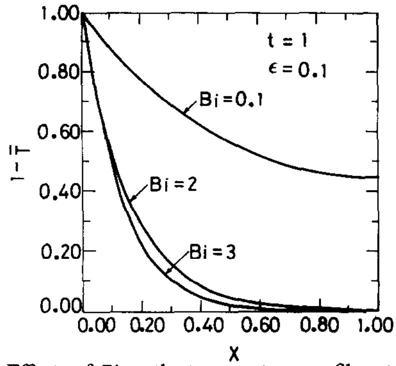

FIG. 6. Effects of Bi on the temperature profiles at $t = 1.0$.

the effects of Biot number on the temperature profiles at different times.

It can be seen that the method presented in this study correctly predicts the temperature profiles for the transient conduction in a 2-D straight fin. The advantage of the proposed method is that it is simple and straightforward. It can easily be extended to study the problem of transient conduction in a multi-layer, unsymmetrical fin. By applying the perturbation technique and the average method proposed in this study, the 2-D problem of transient conduction in a composite fin reduces to that of solving a set of coupled parabolic partial differential equations. After applying the linear operator method, the problem reduces to that of solving a set of coupled Sturm–Liouville equations.

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UNE NOUVELLE APPROCHE DE LA CONDUCTION VARIABLE DANS UNE AILETTE RECTANGULAIRE A 2-D

Résumé—Cette étude développe une méthode de réduction du problème de conduction variable en 2-D dans une ailette rectangulaire, à celui d'un problème 1-D. Après application de la technique de perturbation et d'une méthode de moyenne, le problème 2-D de conduction variable est réduit à 1-D. Une méthode d'opérateur linéaire peut alors être appliquée pour résoudre l'équation aux dérivées partielles, si le problème associé de Sturm–Liouville est traitable. Un calcul des profils de température variable en 2-D dans une ailette rectangulaire confirme la validité de la méthode développée dans ce travail.

EIN NEUES VERFAHREN ZUR BERECHNUNG DER INSTATIONÄREN WÄRMELEITUNG IN EINER ZWEIDIMENSIONALEN RECHTWINKLIGEN RIPPE

Zusammenfassung—Das Ziel der vorliegenden Arbeit ist es, ein Verfahren zu entwickeln, mit dem das Problem der instationären Wärmeleitung in einer zweidimensionalen rechtwinkligen Rippe auf ein eindimensionales Problem zurückgeführt werden kann. Nachdem das Störungsverfahren und eine Methode zur Mittelwertbildung verwendet worden ist, wird das Problem der instationären Wärmeleitung in einer zweidimensionalen rechtwinkligen Rippe auf ein eindimensionales Problem reduziert. Die ermittelte partielle Differentialgleichung kann mit einer linearen Operator-Methode gelöst werden, wenn das damit verbundene Sturm–Liouville-Problem einfach behandelt werden kann. Die vorliegende Berechnung der instationären Temperaturverteilung in einer zweidimensionalen rechtwinkligen Rippe bestätigt die Richtigkeit der in dieser Arbeit entwickelten Methode.

НОВЫЙ ПОДХОД К ОПИСАНИЮ НЕСТАЦИОНАРНОЙ ТЕПЛОПРОВОДНОСТИ ДВУМЕРНОГО ПРЯМОУГОЛЬНОГО РЕБРА

Аннотация—Настоящее исследование предпринято с целью разработки метода, позволяющего свести задачу нестационарной теплопроводности в двумерном прямоугольном ребре к одномерной задаче. Данная цель достигается применением методов возмущения и усреднения. Для решения полученного дифференциального уравнения в частных производных может использоваться метод линейного оператора при условии разрешимости задачи Штурма–Лиувилля. Расчет реальных нестационарных температурных профилей в двумерном прямоугольном ребре подтверждает справедливость предложенного метода.